



Very special divisors on real algebraic curves

Jean-Philippe Monnier

► To cite this version:

| Jean-Philippe Monnier. Very special divisors on real algebraic curves. 2009. hal-00389352

HAL Id: hal-00389352

<https://hal.science/hal-00389352>

Preprint submitted on 28 May 2009

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

Very special divisors on real algebraic curves

Jean-Philippe Monnier

Département de Mathématiques, Université d'Angers,
2, Bd. Lavoisier, 49045 Angers cedex 01, France
e-mail: monnier@tonton.univ-angers.fr

Mathematics subject classification (2000): 14C20, 14H51, 14P25, 14P99

Abstract

We study special linear systems called “very special” whose dimension does not satisfy a Clifford type inequality given by Huisman. We classify all these very special linear systems when they are compounded of an involution. Examples of very special linear systems that are simple are also given.

1 Introduction and preliminaries

1.1 Introduction

In this note, a real algebraic curve X is a smooth proper geometrically integral scheme over \mathbb{R} of dimension 1. A closed point P of X will be called a real point if the residue field at P is \mathbb{R} , and a non-real point if the residue field at P is \mathbb{C} . The set of real points $X(\mathbb{R})$ of X decomposes into finitely many connected components, whose number will be denoted by s . By Harnack’s Theorem ([5, Th. 11.6.2 p. 245]) we know that $s \leq g + 1$, where g is the genus of X . A curve with $g + 1 - k$ real connected components is called an $(M - k)$ -curve. Another topological invariant associated to X is $a(X)$, the number of connected components of $X(\mathbb{C}) \setminus X(\mathbb{R})$ counted modulo 2. The pair $(s, a(X))$ is referred to as the topological type of X . A theorem of Klein asserts that there exists real curves of genus g with topological type (s, a) if and only if the integers g, s and a obey the following restrictions:

Proposition 1.1 [13]

- 1) If $a(X) = 0$, then $1 \leq s \leq g + 1$ and $s \equiv g + 1 \pmod{2}$.
- 2) If $a(X) = 1$, then $0 \leq s \leq g$.

The group $\text{Div}(X)$ of divisors on X is the free abelian group generated by the closed points of X . If D is a divisor on X , we will denote by $\mathcal{O}(D)$ its associated invertible sheaf. The dimension of the space of global sections of this sheaf will be denoted by $\ell(D)$. Since a principal divisor has an even degree on each connected component of $X(\mathbb{R})$ (e.g. [10] Lem. 4.1), the number $\delta(D)$ (resp. $\beta(D)$) of connected components C of $X(\mathbb{R})$ such that the degree of the restriction of D to C is odd (resp. even) is an invariant of the linear system $|D|$ associated to D . If $\ell(D) > 0$, the dimension of the linear system $|D|$ is $\dim |D| = \ell(D) - 1$. Let K be the canonical divisor. If $\ell(K - D) = \dim H^1(X, \mathcal{O}(D)) > 0$, D is said to be special. If not, D is said to be non-special. By Riemann-Roch, if $\deg(D) > 2g - 2$ then D is non-special.

Assume D is effective of degree d . If D is non-special then the dimension of the linear system $|D|$ is given by Riemann-Roch. If D is special, then the dimension of the linear system $|D|$ satisfies

$$\dim |D| \leq \frac{1}{2}d.$$

This is the well known Clifford inequality for complex curves that works also for real curves.

Huisman ([12, Th. 3.2]) has shown that:

Theorem 1.2 *Assume X is an M -curve or an $(M - 1)$ -curve. Let $D \in \text{Div}(X)$ be an effective and special divisor of degree d . Then*

$$\dim |D| \leq \frac{1}{2}(d - \delta(D)).$$

Huisman inequality is not valid for all real curves and the author has obtained the following theorem.

Theorem 1.3 [14, Th. A] *Let D be an effective and special divisor of degree d . Then either*

$$\dim |D| \leq \frac{1}{2}(d - \delta(D)) \quad (\text{Clif1})$$

or

$$\dim |D| \leq \frac{1}{2}(d - \beta(D)) \quad (\text{Clif2})$$

Moreover, D satisfies the inequality (Clif 1) if either $s \leq 1$ or $s \geq g$.

In this note we are interested in special divisors that do not satisfy the inequality (Clif1) given by Huisman.

Definition 1.4 *Let D be an effective and special divisor of degree d . We say that D is a very special divisor (or $|D|$ is a very special linear system) if D does not satisfy the inequality (Clif 1) i.e. $\dim |D| > \frac{1}{2}(d - \delta(D))$.*

In the previous cited paper, the author has obtained a result in this direction.

Theorem 1.5 [14, Th. 2.18] *Let D be a very special and effective divisor of degree d on a real curve X such that (Clif 2) is an equality i.e.*

$$r = \dim |D| = \frac{1}{2}(d - \beta(D)) > \frac{1}{2}(d - \delta(D))$$

then X is an hyperelliptic curve with $\delta(g_2^1) = 2$ and $|D| = rg_2^1$ with r odd.

The author wishes to express his thanks to D. Naie and M. Coppens for several helpful comments concerning the paper.

1.2 Preliminaries

We recall here some classical concepts and notation we will be using throughout the paper.

Let X be a real curve. We will denote by $X_{\mathbb{C}}$ the base extension of X to \mathbb{C} . The group $\text{Div}(X_{\mathbb{C}})$ of divisors on $X_{\mathbb{C}}$ is the free abelian group on the closed points of $X_{\mathbb{C}}$. The Galois group $\text{Gal}(\mathbb{C}/\mathbb{R})$ acts on the complex variety $X_{\mathbb{C}}$ and also on $\text{Div}(X_{\mathbb{C}})$. We will always indicate this action by a bar. Identifying $\text{Div}(X)$ and $\text{Div}(X_{\mathbb{C}})^{\text{Gal}(\mathbb{C}/\mathbb{R})}$, if P is a non-real point of X then $P = Q + \bar{Q}$ with Q a closed point of $X_{\mathbb{C}}$.

Let $D \in \text{Div}(X)$ be a divisor with the property that $\mathcal{O}(D)$ has at least one nonzero global section. The linear system $|D|$ is called base point free if $\ell(D - P) \neq \ell(D)$ for all closed points P of X . If not, we may write $|D| = E + |D'|$ with E a non zero effective divisor called the base divisor of $|D|$, and with $|D'|$ base point free. A closed point P of X is called a base point of $|D|$ if P belongs to the support of the base divisor of $|D|$. We note that

$$\dim |D| = \dim |D'|.$$

As usual, a g_d^r is an r -dimensional complete linear system of degree d on X . Let $|D|$ be a base point free g_d^r on X . The linear system $|D|$ defines a morphism $\varphi : X \rightarrow \mathbb{P}_{\mathbb{R}}^r$ onto a non-degenerate (but maybe singular) curve in $\mathbb{P}_{\mathbb{R}}^r$. If φ is birational (resp. an isomorphism) onto $\varphi(X)$, the g_d^r (or D) is called simple (resp. very ample). Let X' be the normalization of $\varphi(X)$, and assume D is not simple i.e. $|D - P|$ has a base point for any closed point P of X . Thus, the induced morphism $\varphi : X \rightarrow X'$ is a non-trivial covering map of degree $k \geq 2$. In particular, there exists $D' \in \text{Div}(X')$ such that $|D'|$ is a $g_{\frac{d}{k}}^r$ and such that $D = \varphi^*(D')$, i.e. $|D|$ is induced by X' . If g' denote the genus of X' , $|D|$ is classically called compounded of an involution of order k and genus g' . In the case $g' > 0$, we speak of an irrational involution on X .

The reader is referred to [1] and [11] for more details on special divisors. Concerning real curves, the reader may consult [10]. For $a \in \mathbb{R}$ we denote by $[a]$ the integral part of a , i.e. the biggest integer $\leq a$.

2 Non-simple very special divisors

We first characterize the very special pencils.

Proposition 2.1 *Let D be a very special divisor of degree $d > 0$ such that $\dim |D| = 1$. Then $D = P_1 + \dots + P_s$ with P_1, \dots, P_s some real points of X such that no two of them belong to the same connected component of $X(\mathbb{R})$ i.e. $d = \delta(D) = s$. Moreover D is base point free.*

Proof: Since D is special, we may assume that D is effective. Consequently, $d \geq \delta(D)$ and since $\dim |D| = 1 > \frac{1}{2}(d - \delta(D))$ we have $d = \delta(D)$ and $\dim |D| = 1 = \frac{1}{2}(d - \delta(D)) + 1$.

Since $d = \delta(D)$, $D = P_1 + \dots + P_d$ with P_1, \dots, P_d some real points of X such that no two of them belong to the same connected component of $X(\mathbb{R})$.

Assume $d < s$. Choose a real point P in one of the $s - d$ real connected components that do not contain any of the points P_1, \dots, P_d . Since $\ell(D) = 2$ then $\mathcal{O}(D - P)$ has a nonzero global section and $D - P$ should be linearly equivalent to an effective divisor D' of degree $d - 1$ satisfying $\delta(D') = d + 1$. This is impossible.

So $d = s$ and suppose $|D|$ is not base point free. If $|D|$ has a real base point P , then $\dim |D - P| = 1$ and $\deg(D - P) = \delta(D - P) = s - 1$, contradicting the case $d < s$. If $|D|$ has a non-real base point Q , then $\ell(D - Q) > 0$ and $D - Q$ is linearly equivalent to an effective divisor D' of degree $s - 2$ satisfying $\delta(D') = s$, which is again impossible. \square

From the previous proposition, we get the the following corollary:

Corollary 2.2 *Let D be a divisor of degree $d > 0$ such that $D = P_1 + \dots + P_d$ with P_1, \dots, P_d real points of X such that no two of them belong to the same connected component of $X(\mathbb{R})$. Then $\dim |D| = 0$ if $d < s$ and $\dim |D| \leq 1$ if $d = s$.*

The following lemma will allow us to restrict the study to base point free linear systems.

Lemma 2.3 *Let $D \in \text{Div}(X)$ be an effective divisor of degree d . Let E be the base divisor of $|D|$. Let $|D'| = |D - E|$ be the degree d' base point free part of $|D|$. If $\dim |D'| \leq \frac{1}{2}(d' - \delta(D')) + k$ for a non-negative integer k , then $\dim |D| \leq \frac{1}{2}(d - \delta(D)) + k$.*

Proof: Write $D = D' + E$ where E is the base divisor of $|D|$. Assume $D' \in \text{Div}(X)$ is an effective divisor of degree d' satisfying

$$\dim |D'| \leq \frac{1}{2}(d' - \delta(D')) + k$$

for a non-negative integer k . Since $\dim |D| = \dim |D'|$ and E is effective, we have $\delta(D' + E) \leq \delta(D') + \deg(E)$. Then $\dim |D| = \dim |D' + E| \leq \frac{1}{2}(d' - \delta(D')) + k \leq \frac{1}{2}(\deg(D') + \deg(E) - \delta(D') - \deg(E)) + k \leq \frac{1}{2}(\deg(D' + E) - \delta(D' + E)) + k$ proving the lemma. \square

Let D be a special divisor. Recall that $\delta(D) = \delta(K - D)$ and that $\beta(D) = \beta(K - D)$. The next lemma will allow us to study very special divisors of degree $\leq g - 1$.

Lemma 2.4 *Let $D \in \text{Div}(X)$ be an effective and special divisor of degree d . If $\dim |D| = \frac{1}{2}(d - \delta(D)) + k$ for a positive integer k , then $\dim |K - D| = \frac{1}{2}(\deg(K - D) - \delta(K - D)) + k$.*

Proof: It is a straightforward calculation using Riemann-Roch. \square

We can establish one of the main result of the paper.

Theorem 2.5 *Let D be a non-simple very special divisor of degree d . Then*

$$\delta(D) = s$$

and

$$\dim |D| = \frac{1}{2}(d - \delta(D)) + 1.$$

Moreover D is base point free.

Proof: We prove, by induction on $\dim |D|$, the theorem for a base point free non-simple very special divisor.

Let D be a base point free non-simple divisors of degree d such that $\dim |D| = r > \frac{1}{2}(d - \delta(D))$. Since D is special, we may assume D effective.

If $r = 1$, Proposition 2.1 gives the result.

Assume $r > 1$. Consider the map $\varphi : X \rightarrow \mathbb{P}_{\mathbb{R}}^r$ associated to $|D|$. Let X' be the normalization of $\varphi(X)$. Then the induced morphism $\varphi : X \rightarrow X'$ is a non-trivial covering map of degree $t \geq 2$ and there is $D' \in \text{Div}(X')$ such that $|D'|$ is a $g_{\frac{r}{t}}^r$ and such that $D = \varphi^*(D')$.

Assume $\delta(D) < s$. Let P be a point of a connected component of $X(\mathbb{R})$ where the degree of the restriction of D is even. Since $r \geq 1$, we may assume $D - P$ effective. Since P is real, $P' = \varphi(P)$ is real. Let $D_1 = D - \varphi^*(P')$ and denote by $d_1 = d - t$ its degree. Then D_1 is non-simple and effective since $D_1 = \varphi^*(D' - P')$. Moreover $\dim |D_1| = \dim |D - P| = r - 1$. Since $\dim |D - P| = r - 1 = \frac{1}{2}(d - \delta(D)) + 1 - 1 =$

$\frac{1}{2}(d-1-(\delta(D)+1))+1 = \frac{1}{2}(\deg(D-P)-\delta(D-P))+1$, we see that $D-P$ is a very special divisor. By Lemma 2.3, D_1 is also very special. Since D is base point free, then D' is base point free. Choosing P such that $D'-P'$ is base point free then D_1 is base point free since $\dim |D_1| = \varphi^*(D'-P')| = \dim |D'-P'|$. By induction, $\delta(D_1) = s$ and $\dim |D_1| = r-1 = \frac{1}{2}(d_1-s)+1$ i.e.

$$r = \frac{1}{2}(d_1-s)+2. \quad (1)$$

Remark that $d \geq d_1+2$ (φ is non-trivial) and $\delta(D) \leq s-1$. If $\delta(D) = s-1$ then $d \geq d_1+3$, since $\delta(D_1) = s$ and $\deg(\varphi^*(P')) \geq 2$. Hence we get $r > \frac{1}{2}(d-\delta(D)) \geq \frac{1}{2}(d_1+3-\delta(D)) = \frac{1}{2}(d_1+3-s+1) = \frac{1}{2}(d_1-s)+2$ contradicting (1). If $\delta(D) < s-1$, we get $r > \frac{1}{2}(d-\delta(D)) \geq \frac{1}{2}(d_1+2-\delta(D)) \geq \frac{1}{2}(d_1+2-s+2) = \frac{1}{2}(d_1-s)+2$ contradicting (1).

We have just proved that $\delta(D) = s$. Now assume $r \geq \frac{1}{2}(d-s)+2$. Let P be a real point. Let D_1 be the divisor of degree d_1 constructed as in the above proof that $\delta(D) = s$. Since $\dim |D-P| = r-1 \geq \frac{1}{2}(d-s)+2-1 = \frac{1}{2}(d-1-(\delta(D)-1))+1 = \frac{1}{2}(\deg(D-P)-\delta(D-P))+1$, we see that $D-P$ is a very special divisor. By Lemma 2.3, D_1 is also very special. For a general choice of P , D_1 is also base point free. By induction, $\delta(D_1) = s$ and $\dim |D_1| = r-1 = \frac{1}{2}(d_1-s)+1$ since D_1 is non-simple and base point free. Since $d \geq d_1+2$ then $r = \frac{1}{2}(d_1-s)+2 = \frac{1}{2}(d_1+2-s)+1 \leq \frac{1}{2}(d-\delta(D))+1$, impossible.

We have proved the theorem in the case of base point free divisors. Let D be a non-simple very special divisor. If D has base point, with the previous notation, it means that D' has base point. Write $D = D_2 + E$ where E is the base divisor of $|D|$. Since $D = \varphi^*(D')$ and $\dim |D| = \dim |D'|$, we have $E = \varphi^*(E')$ where E' is the base divisor of $|D'|$, it means that D_2 is also non-simple. By Lemma 2.3 and the proof for base point free divisors, $D_2 \in \text{Div}(X)$ is an effective divisor of degree d_2 satisfying $\dim |D_2| = r = \frac{1}{2}(d_2-\delta(D_2))+1$ and $\delta(D_2) = s$. Let e denote the degree of E . Since $\dim |D_2| = \frac{1}{2}(d_2-\delta(D_2))+1$ and since D is very special, we have $r = \frac{1}{2}(d-\delta(D))+1$ by Lemma 2.3. But $r = \frac{1}{2}(d_2-s)+1$, hence $d-d_2 = e = \delta(D)-s \leq 0$. Consequently, $e = 0$ and $\delta(D) = s$, i.e. D is base point free. \square

3 Very special nets

A net is a linear system of dimension 2 i.e. a g_d^2 .

We recall some classical definitions concerning real curves in projective spaces. Let $X \subseteq \mathbb{P}_{\mathbb{R}}^r$, $r \geq 2$, be a real curve. X is non-degenerate if X is not contained in any hyperplane of $\mathbb{P}_{\mathbb{R}}^r$. In what follows, X is supposed to be non-degenerate. Let C be a connected component of $X(\mathbb{R})$. The component C is called a pseudo-line if the canonical class of C is non-trivial in $H_1(\mathbb{P}_{\mathbb{R}}^r(\mathbb{R}), \mathbb{Z}/2)$. Equivalently, C is a pseudo-line if and only if for each real hyperplane H , $H(\mathbb{R})$ intersects C in an odd number of points, when counted with multiplicities (see [12]).

Before looking at very special nets, we need some lemmas concerning morphisms between real curves and very special divisors.

Lemma 3.1 *Let $\varphi : X \rightarrow X'$ be a covering map of degree t between two real curves X and X' .*

- (i) *If P' is a real point of X' then $\varphi^{-1}(P')$ can contain real and non-real points.*
- (ii) *If Q' is a non-real point of X' then $\varphi^{-1}(Q')$ is totally non-real.*

- (iii) The image by φ of a connected component C of $X(\mathbb{R})$ is either a connected component of $X'(\mathbb{R})$, or a compact connected semi-algebraic subset of a connected component C' of $X'(\mathbb{R})$ corresponding topologically to a closed interval of C' .

The proof of the lemma is trivial.

Lemma 3.2 Let $\varphi : X \rightarrow X'$ be a covering map of degree t between two real curves X and X' . Let P be a real point of $X(\mathbb{R})$ contained in a connected component C of $X(\mathbb{R})$. Let C' denote the connected component of $X'(\mathbb{R})$ containing $P' = \varphi(P)$. If $\deg(\varphi^*(P') \cap C)$ is odd then $\varphi(C) = C'$.

Proof: Suppose $\varphi(C)$ is not a connected component of $X'(\mathbb{R})$. Then $\varphi(C)$, corresponds topologically to a closed interval of a connected component of $X'(\mathbb{R})$. Let P'_1 be one of the two end-points of this interval. Let $P_1 \in \varphi^{-1}(P'_1) \cap C$. Then the ramification index e_{P_1} is even since C is clearly on one side of the fiber $\varphi^{-1}(P'_1)$. Hence $\deg(\varphi^*(P'_1) \cap C)$ is even. It is impossible since $\deg(\varphi^*(P'_1) \cap C) = \deg(\varphi^*(P') \cap C) \pmod{2}$. \square

The following lemma is a generalization of a lemma due to Huisman [12].

Lemma 3.3 Let $D \in \text{Div}(X)$ be a divisor of degree d such that $\ell(D) > 0$. Assume that $d + \delta(D) < 2s + 2k$ with $k \in \mathbb{N}$. Then

$$\dim |D| \leq \frac{1}{2}(d - \delta(D)) + k.$$

Proof: We proceed by induction on k . The case $k = 0$ is given by Lemma [12]. So, assume that $k > 0$ and that $d + \delta(D) < 2s + 2k$. Since $\ell(D) > 0$, we may assume that D is effective. If $d + \delta(D) < 2s + 2k - 2$, the proof is done by the induction hypothesis. Since $d = \delta(D) \pmod{2}$, we assume that $d + \delta(D) = 2s + 2k - 2$. Let Q be a non-real point. Since $\deg(D - Q) + \delta(D - Q) < 2s + 2k - 2$ and $\delta(D - Q) = \delta(D)$, if $\ell(D - Q) > 0$ then, using the induction hypothesis, $\dim |D - Q| \leq \frac{1}{2}(\deg(D - Q) - \delta(D - Q)) + k - 1 = \frac{1}{2}(d - 2 - \delta(D)) + k - 1$. Hence $\dim |D| \leq \dim |D - Q| + 2 \leq \frac{1}{2}(d - \delta(D)) + k$. If $\ell(D - Q) = 0$ then $\dim |D| \leq 1 \leq \frac{1}{2}(d - \delta(D)) + k$, since we have $k > 0$ and $d \geq \delta(D)$ (D is effective). \square

The following proposition shows that the excess can be bounded in terms of r for linear systems of dimension r which do not satisfy (Clif 1).

Proposition 3.4 Let D be an effective and special divisor of degree d on a real curve X . Assume that $r = \dim |D| = \frac{1}{2}(d - \delta(D)) + k + 1$. Then $k \leq \left\lfloor \frac{r-1}{2} \right\rfloor$.

Proof: Assume $r = 2n - \varepsilon$ with $\varepsilon \in \{0, 1\}$. We proceed by induction on n .

If $r = 1$, we get $2 = d - \delta(D) + 2k + 2$. Since $d \geq \delta(D)$, we have $k = 0$ and $d = \delta(D)$.

If $r = 2$, we get $4 = d - \delta(D) + 2k + 2$ i.e. $2 = d - \delta(D) + 2k$. If $k \geq 1$, we must have $d = \delta(D)$, hence $d + \delta(D) \leq 2s$. But Lemma 3.3 says that $k = 0$, which is impossible. So $k = 0$ and $d = \delta(D) + 2$.

Assume $n > 1$ and $k > 0$. Choose two real points P_1, P_2 in the same real connected component such that $\dim |D - P_1 - P_2| = r - 2$. Then $r - 2 = 2(n - 1) - \varepsilon = \frac{1}{2}((d - 2) - \delta(D - P_1 - P_2)) + (k - 1) + 1$. By the induction hypothesis, $(k - 1) \leq \left\lfloor \frac{(r - 2) - 1}{2} \right\rfloor$ i.e.

$$k \leq \left\lfloor \frac{r - 1}{2} \right\rfloor.$$

□

The following theorem is the main result of this section.

Theorem 3.5 *If D is a very special divisor then $\dim |D| \neq 2$.*

Proof: Let D be a very special divisor such that $\dim |D| = 2$. By Lemma 2.3, we can assume D is base point free. By Proposition 3.4 we get $2 = \frac{1}{2}(d - \delta(D)) + 1$, i.e. $d = \delta(D) + 2$. By Lemma 3.3, we have $d + \delta(D) \geq 2s$. Hence $\delta(D) \geq s - 1$ and we have two possibilities: either $d = s + 2$ and $\delta(D) = s$, or $d = s + 1$ and $\delta(D) = s - 1$.

First, assume D is simple. In this case, X is mapped birationally by φ —associated to $|D|$ —onto a curve of degree d in $\mathbb{P}_{\mathbb{R}}^2$. By the genus formula,

$$g = \frac{1}{2}(d - 1)(d - 2) - \mu,$$

with μ the multiplicity of the singular locus of $\varphi(X)$. If $d = s + 2$, we know that $\varphi(X)$ has exactly s pseudo-lines. Since any two distinct pseudo-lines of $\varphi(X)$ intersect each other, we have $\mu \geq \frac{1}{2}(s - 1)(s)$. By the genus formula, $g \leq \frac{1}{2}(s + 1)(s) - \frac{1}{2}(s - 1)(s) = s$. Hence X is an M -curve or an $(M - 1)$ -curve and Theorem 1.2 gives a contradiction. If $d = s + 1$, we similarly find $g \leq s - 1$. Hence X is an M -curve and Theorem 1.2 leads again to a contradiction.

Second, assume D is not simple. By Theorem 2.5, $\delta(D) = s$ and $2 = \frac{1}{2}(d - s) + 1$ i.e. $d = s + 2$. Consider the map $f : X \rightarrow \mathbb{P}_{\mathbb{R}}^r$ associated to $|D|$. Let X' be the normalization of $f(X)$. Then the induced morphism $\varphi : X \rightarrow X'$ is a non-trivial covering map of degree $t \geq 2$ and there is $D' \in \text{Div}(X')$ such that $|D'|$ is a $g_{\frac{d}{t}}^r$ and such that $D = \varphi^*(D')$. Let C_1, \dots, C_s denote the connected components of $X(\mathbb{R})$. Since $\dim |D| = 2$, we may assume $D = P_1 + \dots + P_s + R_1 + T_1$ with $P_i, R_i, T_i \in C_i$. Since the support of D is totally real, Lemma 3.1 implies that the support of D' is totally real. By Lemma 3.2, $\varphi(C_i)$ is a connected component of $X'(\mathbb{R})$ for $i \geq 2$.

Suppose $t = 2$. The points P_1, R_1, T_1 verify that no two of them belong to the same fiber over the points of D' because if it is not the case the degree mod 2 of the restriction of the fiber to C_1 is not constant. Hence we may assume that we have 3 points P'_1, R'_1, T'_1 of the support of D' contained in the same connected component C' of $X'(\mathbb{R})$ such that $\varphi^*(P'_1) = P_1 + P_2$, $\varphi^*(R'_1) = R_1 + P_3$, $\varphi^*(T'_1) = T_1 + P_4$. Then by Lemma 3.2, $\varphi(C_1) = \varphi(C_2) = \varphi(C_3) = \varphi(C_4) = C'$ which is clearly impossible since it would imply that P_2, P_3, P_4 belong to the same connected component.

Suppose $t \geq 3$. Arguing similarly to the case $t = 2$, we conclude that $\varphi(P_1) = \varphi(R_1) = \varphi(T_1)$. Hence the degree of the restriction to a connected component of $X(\mathbb{R})$ of a fiber over a point from the support of D' is either empty or of odd degree. By Lemma 3.2, $\varphi(X(\mathbb{R}))$ is a union of connected components of $X'(\mathbb{R})$. Moreover, since $\varphi(P_1) = \varphi(R_1) = \varphi(T_1)$ and since no two points among P_1, \dots, P_s belong to the same connected component of $X(\mathbb{R})$, we get that $\deg(D') = \delta(D')$. The support of D' consists on a single point in each connected component of $\varphi(X(\mathbb{R}))$. Corollary 2.2 says that $\dim |D'| = \dim |D| \leq 1$, which is again impossible. □

4 Construction of the non-simple very special divisors

We recall the definition of a g' -hyperelliptic curve (see [9, p. 249]). A curve X is called g' -hyperelliptic if there exists $\varphi : X \rightarrow X'$ a non-trivial covering map of degree 2 such that the genus of X' is g' . Classical hyperelliptic curves correspond to 0-hyperelliptic curves.

In this section, we prove that the non-simple very special linear systems of dimension $r > 1$ are lying on some “very special” g' -hyperelliptic curves and its converse.

Theorem 4.1 *Let D be a non-simple very special divisor of degree d such that $\dim |D| = r > 1$. Let $\varphi : X \rightarrow X'$ be the non-trivial covering map of degree $t \geq 2$ induced by $|D|$ such that there is $D' \in \text{Div}(X')$ of degree d' such that $|D'|$ is a $g'_{\frac{d}{t}}$ and such that $D = \varphi^*(D')$. Let g' denote the genus of X' and let s' be the number of connected components of $X'(\mathbb{R})$. Then*

- (i) D is base point free, $r = \frac{1}{2}(d - s) + 1$ and $\delta(D) = s$;
- (ii) $t = 2$ i.e. X is a g' -hyperelliptic curve;
- (iii) s is even, $s' = \frac{s}{2}$, $\varphi(X(\mathbb{R})) = X'(\mathbb{R})$, the inverse image by φ of each connected component of $X'(\mathbb{R})$ is a disjoint union of 2 connected components of $X(\mathbb{R})$;
- (iv) r is odd and $\delta(D') = s'$;
- (v) D' is a base point free non-special divisor and X' is an M -curve;
- (vi) D' is linearly equivalent to an effective divisor $P'_1 + \dots + P'_{s'} + R'_{1,2} + \dots + R'_{1,r}$ with $P'_i, R'_{i,j} \in C'_i$ such that $\dim |P'_1 + \dots + P'_{s'}| = 1$ and such that $R'_{1,2}, \dots, R'_{1,r}$ are general in C'_1 , where $C'_1, \dots, C'_{s'} = C'_{\frac{s}{2}}$ denote the connected components of $X'(\mathbb{R})$;
- (vii) $a(X) = 0$ and g is odd;
- (viii) there is a very special pencil on X : $|\varphi^*(P'_1 + \dots + P'_{s'})|$.

Proof: We keep the notation and the hypotheses of the theorem. Theorem 2.5 gives statement (i).

Assume $t \geq 3$. Let Q' be a non-real point of X' . Since $r > 2$ (by Theorem 3.5), we may assume Q' in the support of D' . By Lemma 3.1, $\delta(\varphi^*(Q')) = 0$. Let $D_1 = D - \varphi^*(Q')$ and let d_1 denote its degree. We have $\dim |D_1| = \dim |D' - Q'| = r - 2 = \frac{1}{2}(d - s) + 1 - 2 = \frac{1}{2}(d - 4 - s) + 1 > \frac{1}{2}(d - 2t - s) + 1 = \frac{1}{2}(\deg(D_1) - \delta(D_1)) + 1$. Theorem 2.5 says that this is not possible and then statement (ii) is proved.

Since $t = 2$ then $d = 2d'$. Since $r = \frac{1}{2}(d - s) + 1$ we see that s is even. Suppose $\varphi(X(\mathbb{R})) \neq X'(\mathbb{R})$. Using Lemma 3.2, there exists a real point P' of $X'(\mathbb{R})$ such that $\varphi^{-1}(P')$ is non-real. Let $D_1 = D - \varphi^*((r - 2)P')$ and let d_1 denote its degree. Choosing P' sufficiently general, $\dim |D_1| = \dim |D' - (r - 2)P'| = 2 = \frac{1}{2}(d - s) + 1 - (r - 2) = \frac{1}{2}(d - 2r + 4 - s) + 1 = \frac{1}{2}(\deg(D_1) - \delta(D_1)) + 1$. Theorem 3.5 says that this is not possible, hence $\varphi(X(\mathbb{R})) = X'(\mathbb{R})$.

If C' is a connected component of $X'(\mathbb{R})$, we have two possibilities for the inverse image by φ of C' : either $\varphi^{-1}(C')$ is a disjoint union of 2 connected components of $X(\mathbb{R})$, or $\varphi^{-1}(C')$ is a connected component of $X(\mathbb{R})$. In the second case, choosing

a real point P' of C' , we can prove as above that $D_1 = D - \varphi^*((r-2)P')$ is a very special divisor such that $\dim |D_1| = 2$, contradicting Theorem 3.5. Hence the inverse image by φ of each connected component of $X'(\mathbb{R})$ is a disjoint union of 2 connected components of $X(\mathbb{R})$. Consequently $s' = \frac{s}{2}$ and $\delta(D) = 2\delta(D')$. Since $\delta(D) = s$, we have $\delta(D') = s'$.

We claim that r is odd. Indeed, if r is even, choosing a non-real point Q' of X then we can prove as above that $D_1 = D - \varphi^*(\frac{r-2}{2}Q')$ is a very special divisor such that $\dim |D_1| = 2$, again impossible by Theorem 3.5 establishing the claim.

So r is odd. We can choose general points $R'_{1,2}, \dots, R'_{1,r}$ in C'_1 such that $\ell(D' - (R'_{1,2} + \dots + R'_{1,r})) = 2$. Since $\deg(D' - (R'_{1,2} + \dots + R'_{1,r})) = r-1 + \frac{s}{2} - (r-1) = s'$, we may assume that there are real points $P'_1, \dots, P'_{s'}$ such that $P'_i \in C'_i$, $D' = P'_1 + \dots + P'_{s'} + R'_{1,2} + \dots + R'_{1,r}$ and $\dim |P'_1 + \dots + P'_{s'}| = 1$. Moreover, $|\varphi^*(P'_1 + \dots + P'_{s'})|$ is a very special pencil. We have proved the statements (vi) and (viii).

The divisor D' is non-special. If D' were special, $D'_1 = P'_1 + \dots + P'_{s'} + R'_{1,2}$ would be special as a subdivisor of D' . Moreover $\dim |D'_1| = 2$ by the above construction. But $\dim |D'_1| = \frac{1}{2}(\deg(D'_1) - \delta(D'_1)) + 1$, hence D'_1 would be a very special divisor such that $\dim |D'_1| = 2$, a contradiction.

Since D' is non-special, $\dim |D'| = r = \frac{1}{2}d - \frac{1}{2}s + 1 = d' - s' + 1 = d' - g'$ by Riemann-Roch. Hence $s' = g' + 1$ and X' is an M -curve.

Since X' is an M -curve, $a(X') = 0$ (see Proposition 1.1). Since $\varphi^{-1}(X'(\mathbb{R})) = X(\mathbb{R})$ then $\varphi(X(\mathbb{C}) \setminus X(\mathbb{R})) = X'(\mathbb{C}) \setminus X'(\mathbb{R})$. If $X(\mathbb{C}) \setminus X(\mathbb{R})$ is connected then also $X'(\mathbb{C}) \setminus X'(\mathbb{R})$, impossible. Hence $a(X) = 0$. Since $a(X) = 0$ and s is even then g is odd by Proposition 1.1. \square

Corollary 4.2 *If X has a non-simple very special divisor then $a(X) = 0$.*

Proof: Let D be a non-simple very special divisor of degree d such that $\dim |D| = r$. If $r > 1$, Theorem 4.1 gives the result. If $r = 1$, let $\varphi : X \rightarrow \mathbb{P}_{\mathbb{R}}^1$ be the morphism induced by $|D|$. By Lemma 2.1, we have $\varphi^{-1}(\mathbb{P}_{\mathbb{R}}^1(\mathbb{R})) = X(\mathbb{R})$. Since $a(\mathbb{P}_{\mathbb{R}}^1) = 0$, we easily get $a(X) = 0$. \square

Remark 4.3 By Theorem 4.1, if there is a non-simple very special divisor D on X such that $\dim |D| > 1$ then there is a very special pencil on X . The converse is not true. For example, let X be real trigonal curve such that $\delta(g_3^1) = 3$. By [10, p. 179], such a trigonal curve exists. The g_3^1 is very special and we get $s = 3$ by Proposition 2.1. Since s is odd, Theorem 4.1 says that there is not a non-simple very special divisor D on X such that $\dim |D| > 1$.

Remark 4.4 In the situation of Theorem 4.1, if in addition the genus of X' is 0, then X is an hyperelliptic curve with $\delta(g_2^1) = 2$ and $|D| = rg_2^1$ with r odd. Such hyperelliptic curves exist (see [14, Rem. 2.11]) for any odd genus $g \geq 3$ and such very special divisors have already been studied in this particular case, they correspond to the extremal cases in [14, Th. 2.18].

In the rest of the section, we prove the converse of Theorem 4.1. We first put a name on the curves appearing in Theorem 4.1.

Definition 4.5 *A curve X of genus g is a very special g' -hyperelliptic curve if g is odd and if there exists $\varphi : X \rightarrow X'$ a non-trivial covering map of degree 2 such that X' is an M -curve of genus g' and the inverse image by φ of any connected component of $X'(\mathbb{R})$ is the union of two connected components of $X(\mathbb{R})$ i.e. X satisfies all the topological properties of Theorem 4.1.*

The existence of very special g' -hyperelliptic curves of odd genus g is proved in [3].

Proposition 4.6 [3]

Let g', g be natural numbers, $g \geq 2$. For each odd g , and each g' verifying $2g' + 2 \leq g + 1$, there exists a very special g' -hyperelliptic curve of genus g .

Proof: The existence of very special g' -hyperelliptic curves of odd genus g is not explicitly written in [3]. We explain here how to deduce from [3, Th. 2] and from the proof of [3, Thm. 2] the existence of these curves.

Let g', g be natural numbers, $g \geq 2$. By [3, Th. 2], if $2g' + 2 \leq g + 1$, there exists a g' -hyperelliptic curve X of genus g whose real part has $s = 2g' + 2$ connected components. Furthermore $a(X) = 0$.

Now assume g odd. We have to look at the proof of [3, Th. 2] in order to see if whether or not the curve X built in this proof is very special. The curve X is very special if the number of connected components of $X'(\mathbb{R})$ is $g' + 1$ and if $\varphi : X \rightarrow X'$ has no real branch point. These two conditions are satisfied (see the case (b1) p. 280 of the proof of [3, Th. 2] and see [2] for an explanation of the notations). \square

Since we have the existence of very special g' -hyperelliptic curves of odd genus g verifying $2g' + 2 \leq g + 1$, we prove now the converse of Theorem 4.1 on these very special curves.

Proposition 4.7 Let X be a very special g' -hyperelliptic curves of odd genus g such that $2g' + 2 \leq g$. Let $\varphi : X \rightarrow X'$ denote the corresponding non-trivial covering map of degree 2. Let $C'_1, \dots, C'_{g'+1}$ denote the connected components of $X'(\mathbb{R})$. Let $P'_1, \dots, P'_{g'+1}$ be some real points of $X'(\mathbb{R})$ such that $P'_i \in C'_i$. If $r \geq 2$, let $R'_{1,2}, \dots, R'_{1,r}$ be general points in C'_1 . We set $D'_1 = P'_1 + \dots + P'_{g'+1}$, $D'_r = P'_1 + \dots + P'_{g'+1} + R'_{1,2} + \dots + R'_{1,r}$ for $r \geq 2$, and $D_r = \varphi^*(D'_r)$. For any odd r such that

$$1 \leq r \leq g - 2g' - 1$$

choosing the general points $R'_{1,2}, \dots, R'_{1,r}$ such that the two real points of $\varphi^{-1}(R'_{1,j})$ are not base points of $K - D_{j-1}$, $2 \leq j \leq r$, then D_r is a non-simple very special divisor such that $\dim|D_r| = r$.

Proof: If $g' = 0$ i.e. X is hyperelliptic and φ is the hyperelliptic map, then [14, Prop. 2.10] gives the result for any choice of $R'_{1,2}, \dots, R'_{1,r}$.

For the rest of the proof, we assume $g' > 0$.

Claim 1: $|D_1|$ is a very special pencil since $2g' + 2 \leq g$.

We consider the linear system $|D_1|$. By Riemann-Roch, we have $\dim|D'_1| \geq 1$ hence $\dim|D_1| \geq 1$. In fact $\dim|D_1| = 1$ by Corollary 2.2 and since the support of D_1 consists of exactly one point in each connected component of $X(\mathbb{R})$. If D_1 is non-special, then

$$\dim|D_1| = 1 = s - g = 2g' + 2 - g$$

by Riemann-Roch. We get a contradiction with the hypothesis $2g' + 2 \leq g$.

Claim 2: If $2 \leq r \leq g - 2g' - 1$ then D_r is special for any choice of $R'_{1,2}, \dots, R'_{1,r}$.

By Riemann-Roch $\dim|D_r| \geq \dim|D'_r| \geq r$. Hence $\dim|D_r| = r + l$ with an integer $l \geq 0$. Assume D_r non-special. We get $r + l = 2g' + 2r - g$ by Riemann-Roch, i.e. $r = g - 2g' + l$, a contradiction.

Now we prove by induction on $r \geq 1$ that $\dim|D_r| = r$.

If $r = 1$, Claim 1 gives the result. Now suppose $2 \leq r+1 \leq g - 2g' - 1$. By Claim 2, D_{r+1} is special. Notice that D_r is also special since D_r is an effective subdivisor of D_{r+1} . By the induction hypothesis, we get $\dim|D_r| = r$. Let $\varphi^*(R'_{1,r+1}) = R_1 + R_2$ then $D_{r+1} = D_r + R_1 + R_2$. Assume $\dim|D_{r+1}| > r+1$ then clearly $\dim|D_{r+1}| = r+2$. Moreover $\dim|D_r + R_1| = r+1$ and $D_r + R_1$ is special since it is an effective subdivisor of D_{r+1} . Let K denote the canonical divisor of X . Since $\dim|D_r + R_1| = \dim|D_r| + 1$, then R_1 is a base point of $|K - D_r|$ contradicting the general choice of $R'_{1,r+1}$ and finishing the induction.

If

$$1 \leq r \leq g - 2g' - 1,$$

then, by Claim 1, Claim 2 and the induction argument with the points $R'_{1,2}, \dots, R'_{1,r}$ general, D_r is special and non-simple such that $\dim|D_r| = r$. If r is odd, it is easy to see that $\dim|D_r| = \frac{1}{2}(\deg(D_r) - \delta(D_r)) + 1$ and the proof is done. \square

Coppens has made the following remark concerning a consequence of Proposition 4.7.

Corollary 4.8 *For any odd $g \geq 2$ and any even $0 < d < g$, there exists a real curve X of genus g with a very special pencil of degree d .*

One interesting question of Coppens is whether the result of Corollary 4.8 is also valid for even genus. Using a result in [4], we give a partial answer to that question.

Proposition 4.9 *Let $g \geq 2$. For any $0 < d < g$ such that there exists an integer $k \geq 2$ with $g = (k-1)(d-1)$, there exists a real curve X of genus g with a very special pencil of degree d .*

Proof: If

$$g - 1 - (k-1)d + k = 0$$

i.e. if $g = (k-1)(d-1)$, by [4][Prop. 2], there exists a real curve X of genus g with d real connected components with a cyclic morphism $f : X \rightarrow \mathbb{P}_{\mathbb{R}}^1$ having only ramification points of index d over k non-real points of $\mathbb{P}_{\mathbb{R}}^1$. Clearly, f corresponds to a very special pencil of degree d . \square

A consequence of the previous Proposition is a different proof of a Gross and Harris [10, p. 179] result mentioned in Remark 4.3 of the previous section, concerning the existence of trigonal curves with a very special pencil.

Corollary 4.10 *For every even $g \geq 4$, there exists a real trigonal curves of genus g with a g_3^1 very special.*

5 Simple very special divisors and very special curves in some projective spaces

In this section, we prove the existence of simple very special divisors.

Proposition 5.1 *Let X be a real trigonal curve and let D be a divisor on X such that:*

(i) $|D| = g_3^1$,

(ii) $\delta(g_3^1) = 3$,

(iii) $g \geq 5$.

Then, the base point free part D' of $K - D$ is a simple very special divisor with $\dim |D'| = g - 3$. Moreover g is even and $s = 3$.

Proof: As we have already noticed in Corollary 4.10 such a trigonal curve with $\delta(g_3^1) = 3$ exists. Since $g \geq 5$, the g_3^1 is unique. By Proposition 2.1, we get $s = 3$. Since $a(X) = 0$, we see that g is even. If $|D| = g_3^1$ then D is a non-simple very special divisor. Let D' be the base point free part of $K - D$. Since $\ell(K - D') \geq \ell(D) = 2$, D' is a special divisor. By Lemmas 2.4 and 2.3, D' is also very special. By Riemann-Roch

$$\dim |D'| = \dim |K - D| = (2g - 5) - g + 2 = g - 3 \geq 2.$$

Since s is odd, Theorem 4.1 forces D' to be simple. \square

Let $X \subseteq \mathbb{P}_{\mathbb{R}}^r$, $r \geq 2$, be a smooth real curve. We assume, in what follows, that X is non-degenerate. We say that X is special (resp. very special) if the divisor associated to the sheaf of hyperplane sections $\mathcal{O}_X(1)$ is special (resp. very special).

Corollary 5.2 *For every odd $r \geq 3$, there exists a very special curve in $\mathbb{P}_{\mathbb{R}}^r$.*

Proof: Let $r \geq 3$ be an odd integer. Let X be a real trigonal curve and let D be a divisor on X such that:

(i) $|D| = g_3^1$,

(ii) $\delta(g_3^1) = 3$,

(iii) $g = r + 3$.

By Proposition 5.1 the base point free part D' of $K - D$ is a simple very special divisor with $\dim |D'| = r$. Hence $\varphi_{|D'|}(X)$ is a very special curve in $\mathbb{P}_{\mathbb{R}}^r$. \square

References

- [1] E. Arbarello, M. Cornalba, P.A. Griffiths, J. Harris, *Geometry of Algebraic Curves*, Grundlehren der mathematischen Wissenschaften 267, Springer-Verlag, New York-Berlin-Heidelberg-Tokyo 1985
- [2] E. Bujalance, F.J. Cirre, J.M. Gamboa, *Double covering of hyperelliptic real algebraic curves*, Journal of Pure and Applied Algebra, 212, 2011-2026, 2008
- [3] E. Bujalance, J.J. Etayo, J.M. Gamboa, *Topological Types of p -Hyperelliptic Real Algebraic Curves*, Math. Z., 194, 275-283, 1987
- [4] E. Bujalance, J.J. Etayo, J.M. Gamboa, G. Martens, *Minimal genus of Klein surfaces admitting an automorphism of given order*, Arch. Math., 52, 191-202, 1989
- [5] J. Bochnak, M. Coste, M-F. Roy, *Géométrie algébrique réelle*, Ergeb. Math. Grenzgeb., 3. Folge, 12. Berlin Heidelberg New York: Springer 1987

- [6] M. Coppens, C. Keem, G. Martens, *Primitive linear series on curves*, Manuscripta Mathematica, 77, 237-264, 1992
- [7] M. Coppens, G. Martens, *Secant space and Clifford's theorem*, Compositio Mathematica, 78, 193-212, 1991
- [8] D. Eisenbud, H. Lange, G. Martens, F-O. Schreyer, *The Clifford dimension of a projective curve*, Compositio Mathematica, 72, 173-204, 1989
- [9] H. M. Farkas, I. Kra, *Riemann Surfaces*, Springer-Verlag, New York-Berlin-Heidelberg 1980
- [10] B. H. Gross, J. Harris, *Real algebraic curves*, Ann. scient. Ec. Norm. Sup. 4^e série, 14, 157-182, 1981
- [11] R. Hartshorne, *Algebraic geometry*, Grad. Texts in Math. 52, Springer Verlag, 1977
- [12] J. Huisman, *Clifford's inequality for real algebraic curves*, Indag. Math., 14 (2), 197-205, 2003
- [13] F. Klein, *Ueber Realitätsverhältnisse bei der einem beliebigen Geschlechte zugehörigen Normalcurve der φ* , Math. Ann., 42, 1-29, 1893
- [14] J. P. Monnier, *Clifford Theorem for real algebraic curves*, To appear in Ann. Inst. Fourier, 2008